Electrostatic Energy and Capacitance

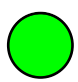
Chapter 26

Energy of a Charge Distribution

How much energy (≡ work) is required to assemble a charge distribution ?.

CASE I: Two Charges

Bringing the first charge does not require energy (≡ work)



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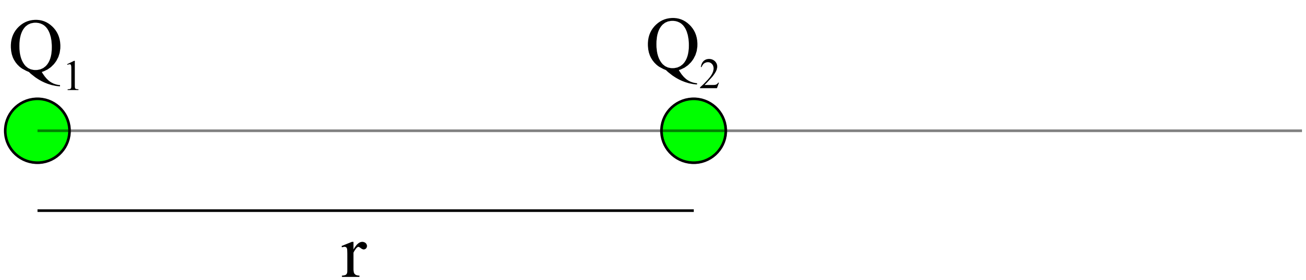

Bringing the second charge requires to perform work against the field of the first charge.

Energy of a Charge Distribution

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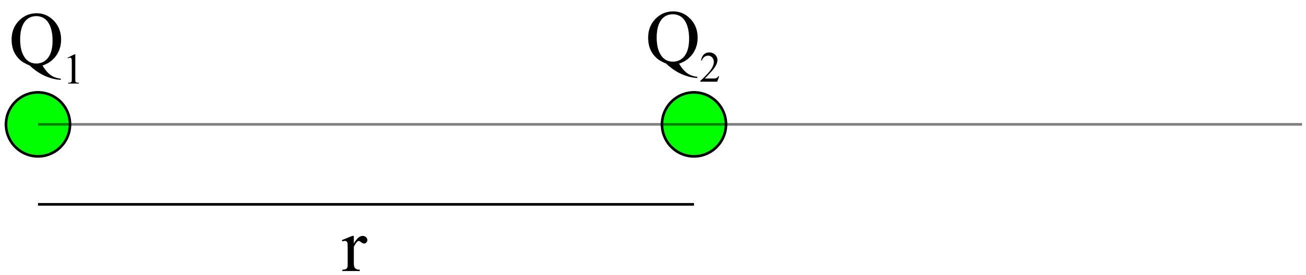


Bringing the second charge requires to perform work against the field of the first charge.

$$W = Q_2 V_1 \quad \text{with} \quad V_1 = (1/4\pi\varepsilon_0)\,(Q_1/r)$$

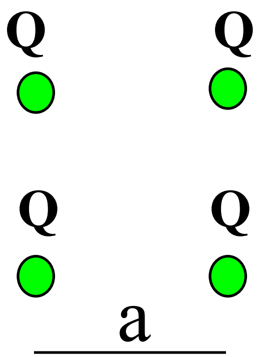
$$\Rightarrow W = (1/4\pi\varepsilon_0)\,(Q_1 Q_2/r) = U$$

$$U = (1/4\pi\varepsilon_0)\,(Q_1 Q_2/r)$$

U = potential energy of two point charges

Energy of a Charge Distribution

CASE II: Several Charges



How much energy is stored in this square charge distribution?, or …
What is the electrostatic potential energy of the distribution?, or …
How much work is needed to assemble this charge distribution?

To answer it is necessary to add up the potential energy of
each pair of charges $\Rightarrow$ $U = \sum U_{ij}$

$$U_{12} = (1/4\pi\varepsilon_0)\,(Q_1\,Q_2\,/r)$$

U_{12} = potential energy of a pair of point charges

Large, Charged, Conducting Plane

Consider a large charged conducting plane: $\sigma = Q/A$

Find E using Gauss's law.

E points straight away, both above and below the surface.

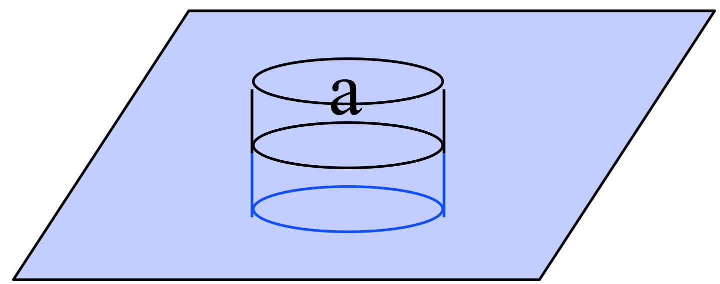
Hence $\Phi = 2Ea$, while $Q_{enc} = \sigma\ a$.

Gauss's law says $\Phi = Q_{enc}/\varepsilon_0$.

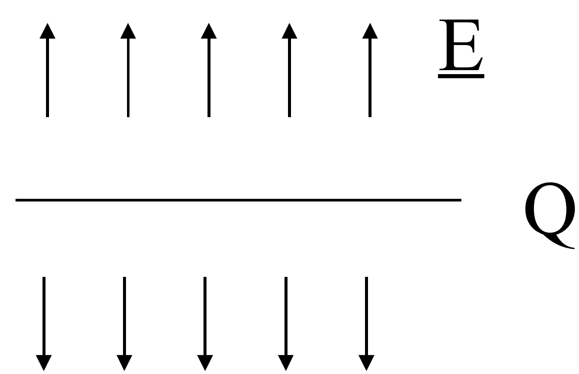
$2Ea = \sigma\ a / \varepsilon_0$ but $\sigma = Q/A$

$$\Rightarrow E = \sigma / 2\varepsilon_0 = Q / 2A\varepsilon_0$$

σ = charge / unit area

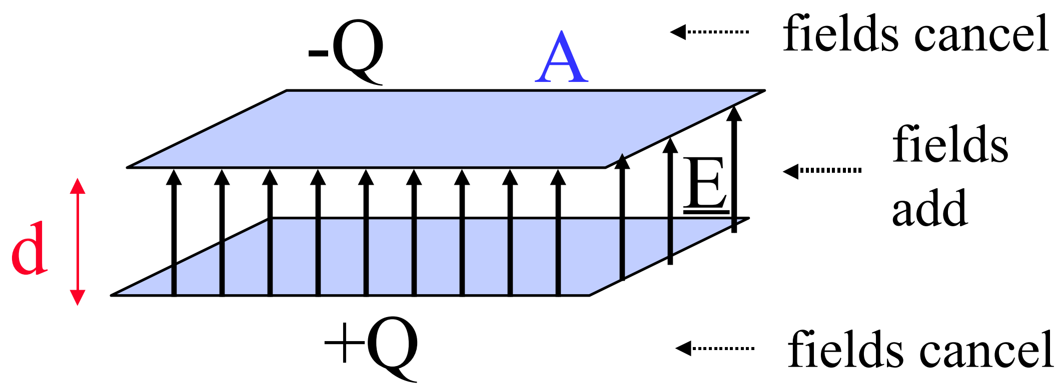


Area A, Charge Q



Energy of a Charge Distribution

CASE III: Parallel Plate Capacitor

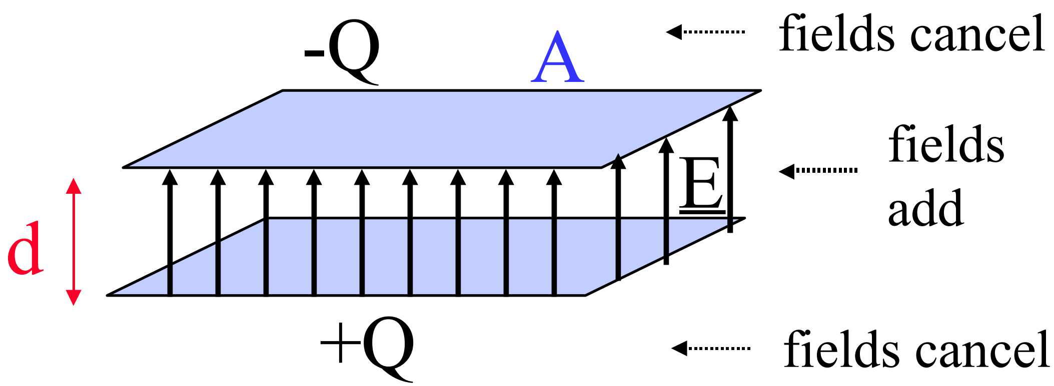


Electric Field $\Rightarrow$ $E = \sigma / \varepsilon_0 = Q / \varepsilon_0 A \quad (\sigma = Q / A)$

Potential Difference $\Rightarrow$ $V = E\, d = Q\, d / \varepsilon_0 A$

Energy of a Charge Distribution

CASE III: Parallel Plate Capacitor



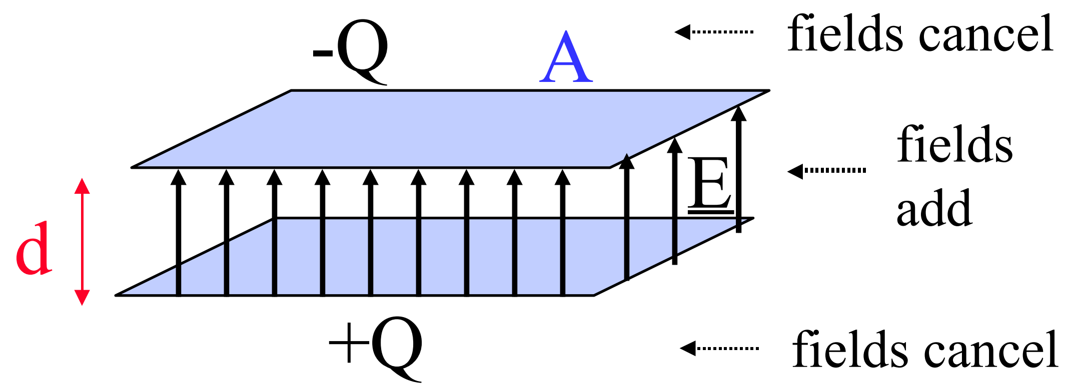
Now, suppose moving an additional very small positive charge dq from the negative to the positive plate. We need to do work. How much work?

$$dW = V\, dq = (q\, d / \varepsilon_0 A)\, dq$$

We can use this expression to calculate the total work needed to
charge the plates to Q, -Q

Energy of a Charge Distribution

CASE III: Parallel Plate Capacitor



$$dW = V\,dq = (q\,d / \varepsilon_0 A)\,dq$$

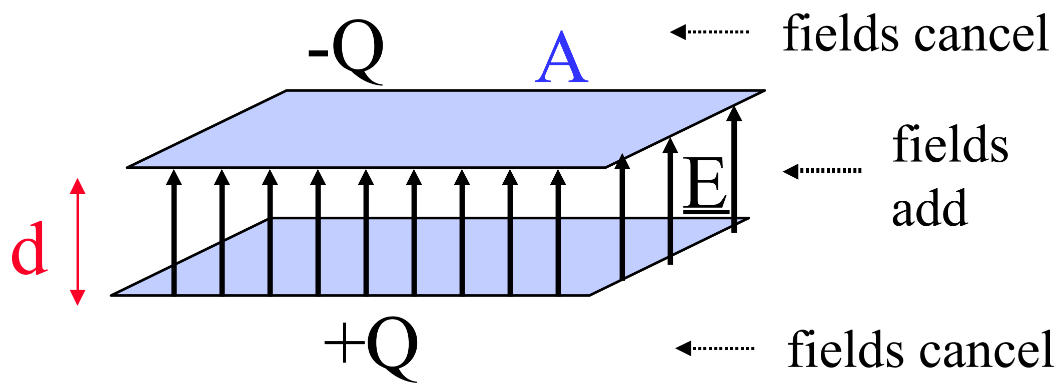
The total work needed to charge the plates to Q, -Q, is given by:

$$W = \int dW = \int (q\,d / \varepsilon_0 A)\,dq = (d / \varepsilon_0 A) \int q\,dq$$

$$W = (d / \varepsilon_0 A)\,[Q^2 / 2] = d\,Q^2 / 2\,\varepsilon_0 A$$

Energy of a Charge Distribution

CASE III: Parallel Plate Capacitor



The work done in charging the plates ends up as stored potential energy of the final charge distribution

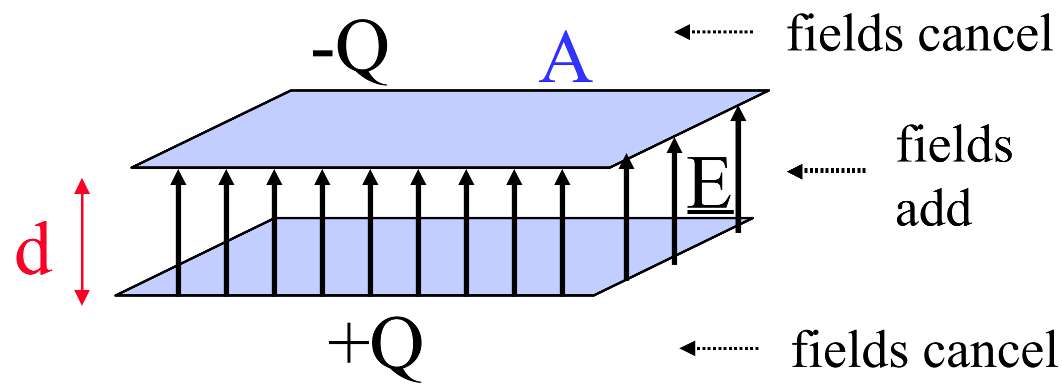
$$W = U = d\,Q^2 / 2\,\varepsilon_0\,A$$

Where is the energy stored ?

The energy is stored in the electric field

Energy of a Charge Distribution

CASE III: Parallel Plate Capacitor



The energy U is stored in the field, in the region between the plates.

$$U = d\,Q^2 / 2\,\varepsilon_0\,A = (1/2)\,\varepsilon_0\,E^2\,A\,d$$

$$E = Q / (\varepsilon_0\,A)$$

The volume of this region is **Vol = A d**, so we can define the **energy density u_E** as:

$$u_E = U / A\,d = (1/2)\,\varepsilon_0\,E^2$$

Energy of a Charge Distribution

CASE IV: Arbitrary Charge Distribution

$$u_E = U / A\,d = (1/2)\,\varepsilon_0\,E^2$$

Electric Energy Density

Although we derived this expression for the uniform field of a parallel plate capacitor, this is a universal expression valid for any electric field.

When we have an arbitrary charge distribution, we can use $\mathbf{u_E}$ to calculate the stored energy **U**

$$\mathbf{dU = u_E\,d(Vol) = (1/2)\,\varepsilon_0\,E^2\,d(Vol) \Rightarrow U = (1/2)\,\varepsilon_0 \int E^2\,d(Vol)}$$

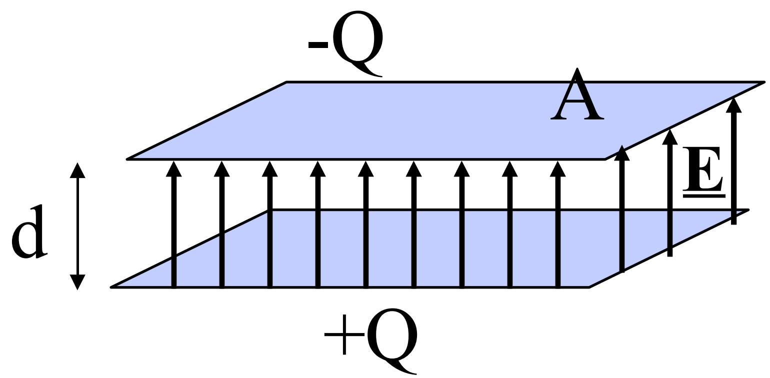
[The integral covers the entire region in which the field E exists]

A Shrinking Sphere

A sphere of radius R_1 carries a total charge Q distributed evenly over its surface. How much work does it take to shrink the sphere to a smaller radius R_2 ?.

Capacitance

Two parallel plates charged Q and –Q respectively constitute a **capacitor**



The electric field between the plates is $\mathbf{E = Q / A\, \varepsilon_0}$

The potential difference between the plates is $\mathbf{V = E\, d = Q\, d / A\, \varepsilon_o}$

$\Rightarrow$ The charge Q gives rise to a potential difference V such that

$$V = Q\, d / A\, \varepsilon_0$$

The ratio $C = Q / V = A\, \varepsilon_0 / d$ is called the **capacitance**

$$\mathbf{C = Q / V}$$

[Units: Coulomb /Volt = Farad]

Capacitance

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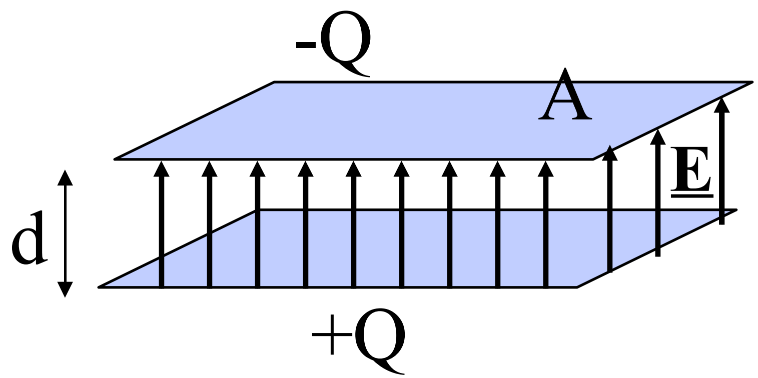

C = Q / V

The relationship C = Q / V is valid for any charge configuration (Indeed this is the definition of capacitance or electric capacity)

In the particular case of a parallel plate capacitor

$$\mathbf{C = Q / V = \varepsilon_0 A / d}$$

The capacitance is directly proportional to the area of the plates and inversely proportional to the separation between the plates

What Does a Capacitor Do?

- Stores electrical charge.
- Stores electrical energy.

Capacitors are basic elements of electrical circuits both macroscopic (as discrete elements) and microscopic (as parts of integrated circuits).

Capacitors are used when a sudden release of energy is needed (such as in a photographic flash).

Electrodes with capacitor-like configurations are used to control charged particle beams (ions, electrons).

What Does a Capacitor Do?

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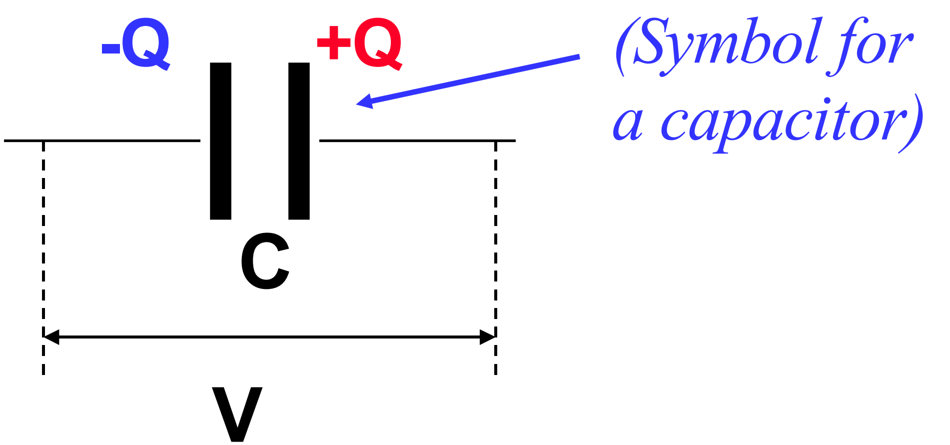
The charge is easy to see. If a certain potential, V, is applied to a capacitor C, it must store a charge **$Q=CV$**:



What Does a Capacitor Do?

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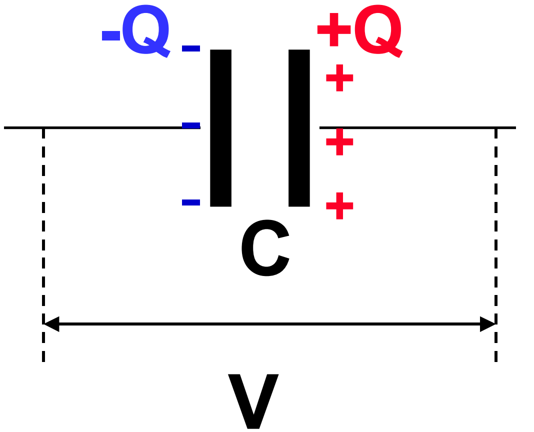
An electrical field is created between the plates

$\Rightarrow$ an energy $U = (1/2)\, \varepsilon_0\, E^2\, A\, d = d\, Q^2 / 2\, \varepsilon_0\, A$

is stored between the plates of the capacitor



Energy Stored in a Capacitor

$$U = (1/2)\ \varepsilon_0\ E^2\ A\ d = d\ Q^2 / 2\ \varepsilon_0\ A$$

$$\text{But } C = \varepsilon_0\ A / d$$

Then: $U = (1/2)\ Q^2 / C = (1/2)\ C\ V^2$ **Potential Energy**

And: $u = (1/2)\ \varepsilon_0\ E^2$ **Potential Energy Density**

Although we calculated **U** and **u** for the parallel plate capacitor, the expressions obtained are valid for any geometry.

Energy Stored in a Capacitor

- Suppose we have a capacitor with charge q (+ and -).
- Then we increase the charge by dq (+ and -).
- We must do work $dW = Vdq$ to increase charge:

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- Integrating q from 0 to Q, we can find the total stored (potential) electric energy:

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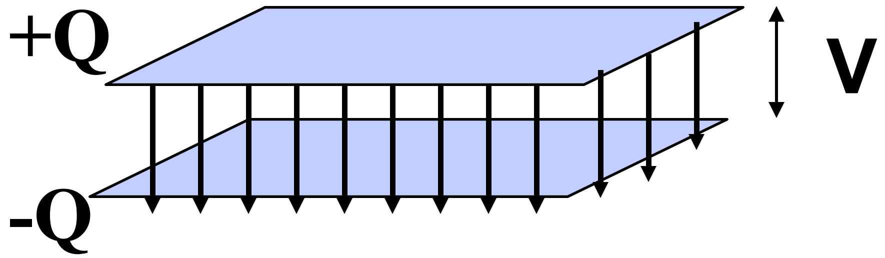
$$U = W = \frac{Q^2}{2C} = \frac{1}{2} CV^2$$

Energy Density.

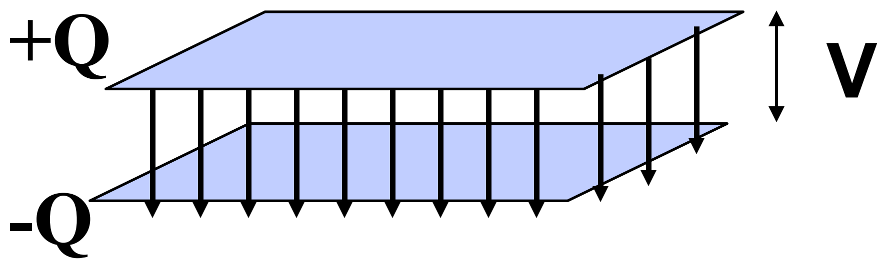


- Now compute the energy density, u_E, inside the capacitor.
- For a parallel plate capacitor of volume A.d,

$$u_E = U/(Ad) = (1/2\ CV^2)/Ad$$

But for a parallel plate capacitor , $C = \varepsilon_0 A/d$

Energy Density.



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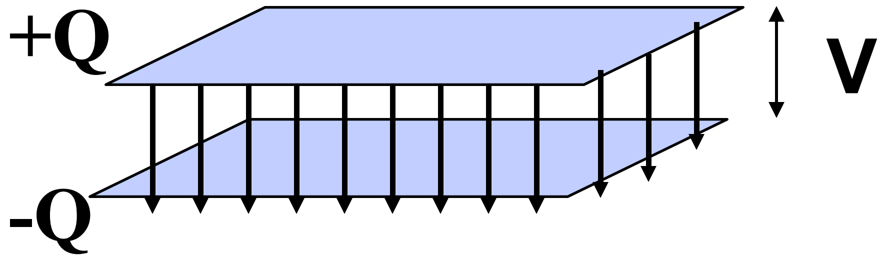
$$\rightarrow \quad u_E = (\varepsilon_0/2)(V/d)^2 = (\varepsilon_0/2)E^2$$

Energy Density.



$$u_E = (\varepsilon_0/2)(V/d)^2 = (\varepsilon_0/2)E^2$$

- This leads to another understanding of electric field
- The energy is stored in the FIELD, rather than in the plates!
- If an electric field exists, then you can associate an electric potential energy of $(\varepsilon_0/2)E^2$

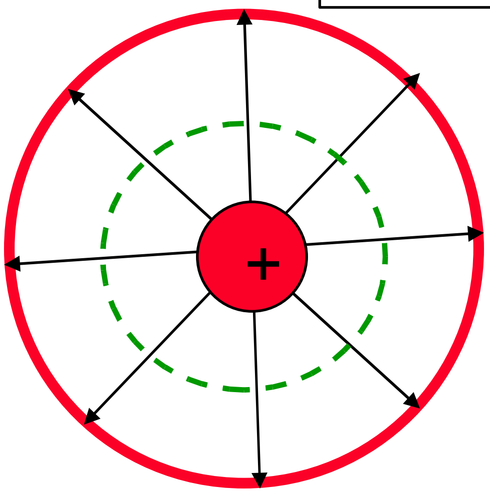
Cylindrical Capacitor

Construct Gaussian surface around inner cylinder. - - - - -

Then $q = \varepsilon_0 EA = \varepsilon_0 E\,(2\pi rL)$

Therefore $E = q/(\varepsilon_0 2\pi rL)$

inner rad. a
outer rad. b
length L



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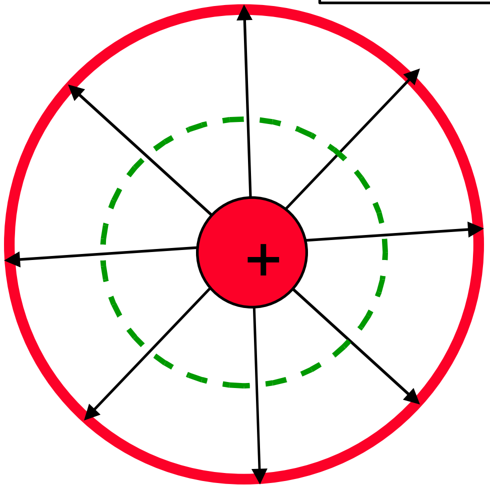
$$V = -\int_i^f \vec{E} \bullet d\vec{r} = \int_+^- E\,dr$$

$$= \frac{q}{2\pi\varepsilon_0 L}\int_a^b \frac{dr}{r} = \frac{q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$$



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$$C = 2\pi\varepsilon_0 \frac{L}{\ln\left(b/a\right)}$$ in Farads

inner rad. a
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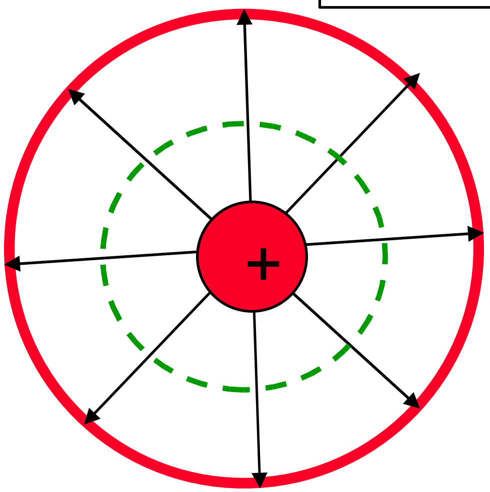

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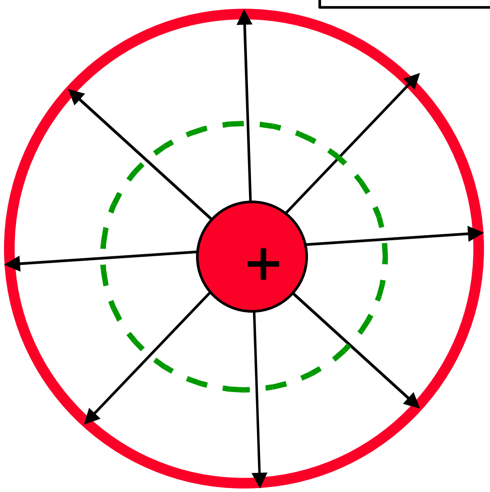
inner rad. a
outer rad. b
length L



Note: C is made larger by making (b - a) as small as possible.

Energy Stored in a Cylindrical Capacitor

The energy stored in a capacitor C (with charge Q and voltage difference V) is given by:

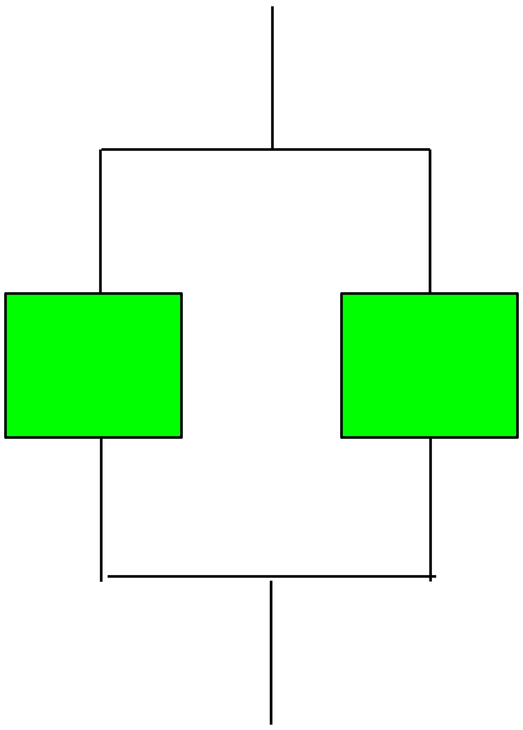
$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

Using $C = 2\pi\varepsilon_0 L / \ln(b/a)$ we obtain:

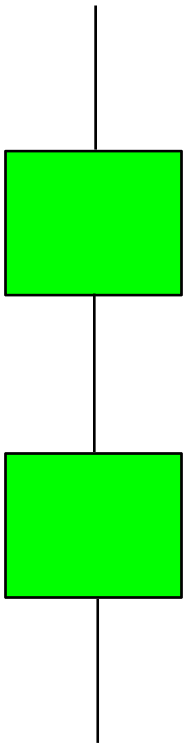
$$U = q^2 \ln(b/a) / 4\pi\varepsilon_0 L$$

The energy **U** is stored in the electric field between the two conductors

Parallel and Series

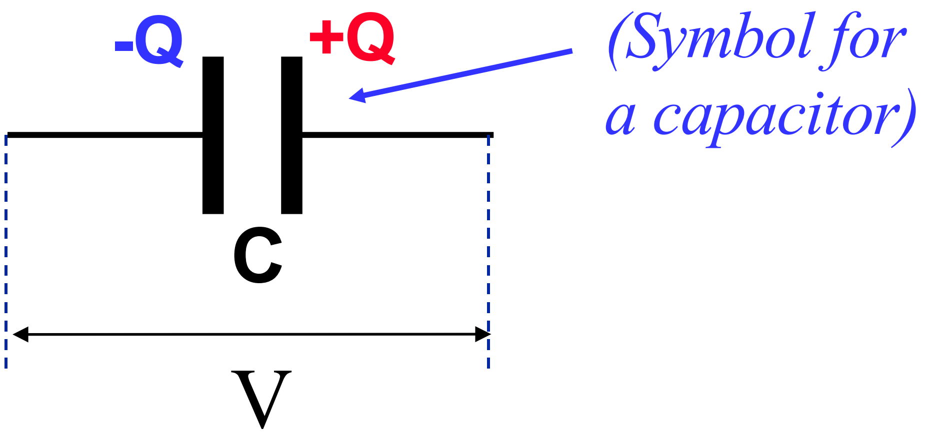


Parallel



Series

Capacitors in Circuits

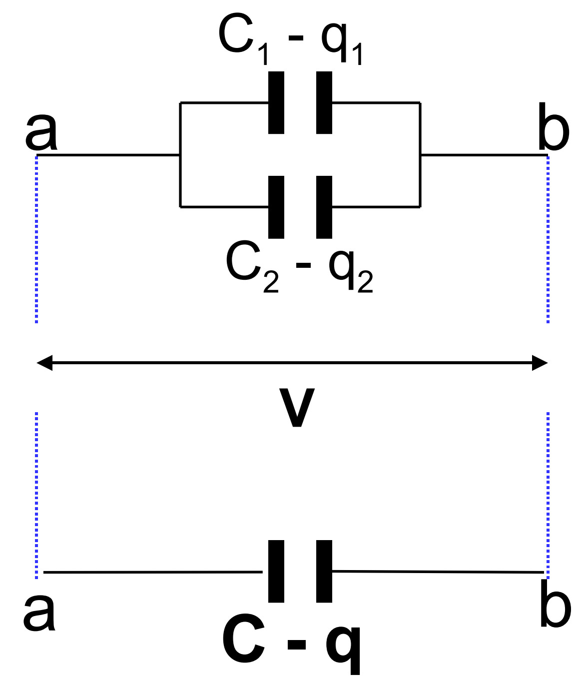


A piece of metal in equilibrium has a constant value of potential.

Thus, the potential of a plate and attached wire is the same.

The potential difference between the ends of the wires is V, the same as the potential difference between the plates.

Capacitors in Parallel

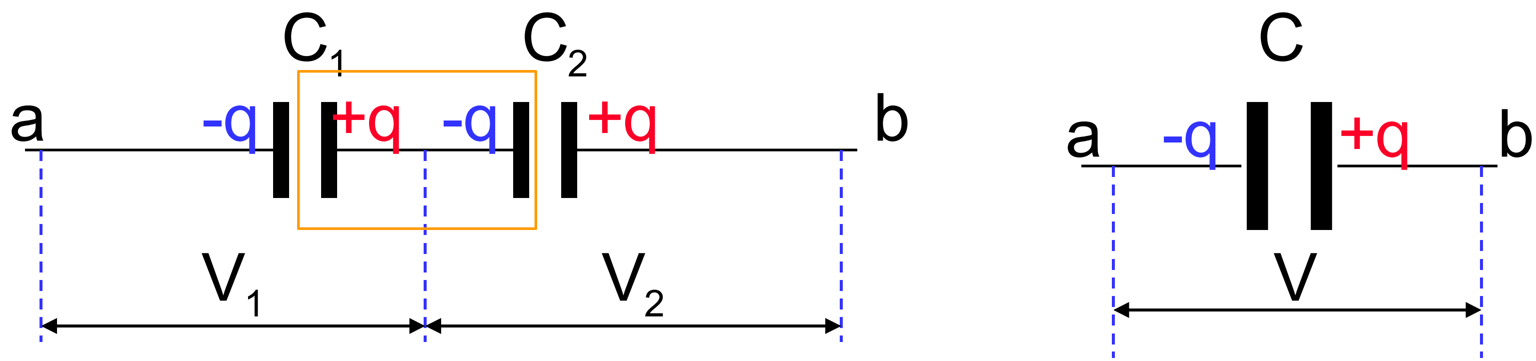


- Suppose there is a potential difference V between a and b.
- Then $q_1 V = C_1$ & $q_2 V = C_2$
- We want to replace C_1 and C_2 with an equivalent capacitance $C = q V$
- The charge on C is $q = q_1 + q_2$
- Then $C = q V = (q_1 + q_2) V = q_1 V + q_2 V = C_1 + C_2$

$$C = C_1 + C_2$$

- This is the equation for capacitors in **parallel**.
- Increasing the number of capacitors increases the capacitance.

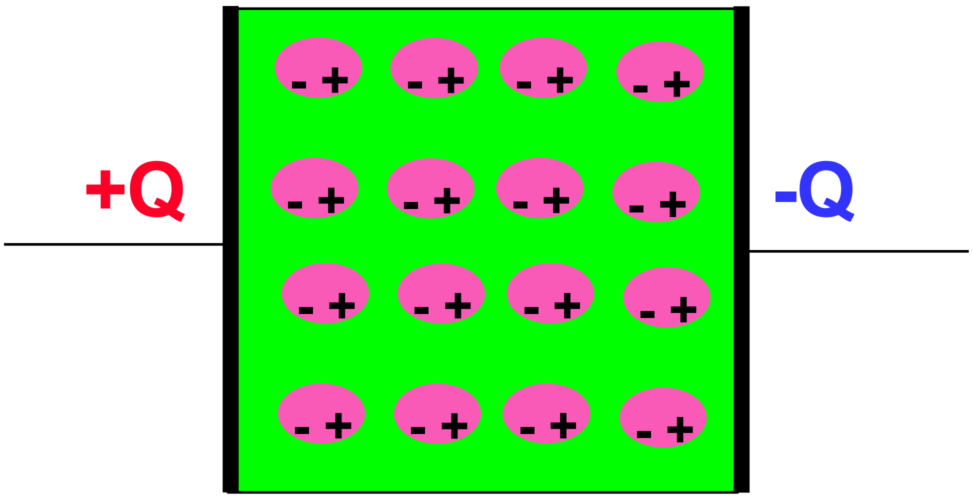
Capacitors in Series



- Here the total potential difference between a and b is $V = V_1 + V_2$
- Also $V_1 = (1/C_1)\, q$ and $V_2 = (1/C_2)\, q$
- The charge on every plate (C_1 and C_2) must be the same (in magnitude)
- Then: $V = V_1 + V_2 = q / C_1 + q / C_2 = [(1/C_1) + (1/C_2)]\, q$
- or, $V = (1/C)\, q \quad \Rightarrow$ $$\mathbf{1 / C = 1 / C_1 + 1 / C_2}$$
- This is the equation for capacitors in **series**.
- Increasing the number of capacitors decreases the

Dielectrics in Capacitors

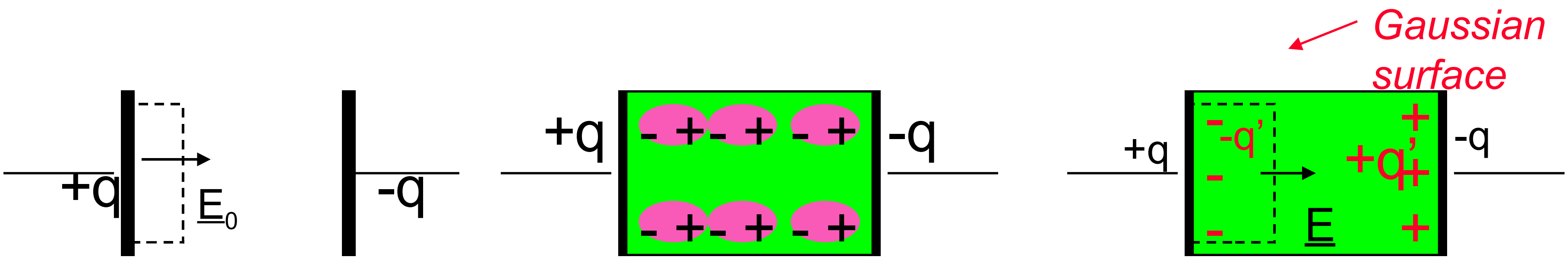
- Suppose we fill the space between the plates of a capacitor with an insulating material (a "dielectric"):



- The material will be "polarized" - electrons are pulled away from atom cores

- Consequently the E field within the capacitor will be reduced

Dielectrics in Capacitors



- We calculate the new field E using Gauss' Law, and noting there is an induced charge q' on the surface of the dielectric.

- $\mathbf{E_0 = q / (\varepsilon_0 A)}$ **and** $\mathbf{E = (q - q') / (\varepsilon_0 A)}$
- $\mathbf{E_0 / E = q / (q - q') \Rightarrow E_0 / E = \kappa}$ **or** $\mathbf{E = E_0/\kappa}$
- **The field is reduced by factor κ** $\mathbf{\Rightarrow \quad E = E_0/\kappa}$

- **The constant κ is called the Dielectric Constant**

$$\kappa = q / (q - q') = 1 / [1-(q'/q)] > 1$$

Effect on Capacitance

- A dielectric reduces the electric field by a factor κ ($E=E_0/\kappa$)
- Hence $V = E\,d$ is reduced by κ [$V = (E_0/\kappa)\,d = V_0/\kappa$)]
- and $C= Q/V$ is increased by κ [$C = (Q\,\kappa) / V_0 = C_0\,\kappa$)

$$\therefore \quad C = \frac{\varepsilon_o \kappa \quad A}{d}$$

parallel plate capacitor with dielectric.

- Adding a dielectric ***increases*** the capacitance.

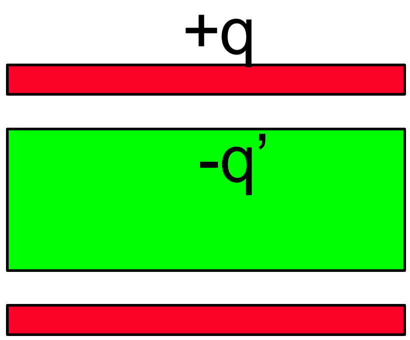
Dielectrics & Gauss's Law

With a dielectric present, Gauss's Law can be rewritten from

$$\varepsilon_0 \oint \underline{E} \bullet \underline{dA} = q - q'$$

to

$$\varepsilon_0 \oint \kappa_e \underline{E} \bullet \underline{dA} = q$$



Instead of having to think about the confusing induced charge q', we can simply use the free charge q. But E is replaced by $\kappa_e E$.